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Review Article

# A brief history of edge-colorings – with personal reminiscences\*

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## Abstract

In this article we survey some important milestones in the history of edge-colorings of graphs, from the earliest contributions of Peter Guthrie Tait and Dénes König to very recent work.

**Keywords:** edge-coloring; graph theory history; Frank Harary.

**2020 Mathematics Subject Classification:** 01A60, 05-03, 05C15.

## 1. Introduction

We begin with some basic remarks. If  $G$  is a graph, then its *chromatic index* or *edge-chromatic number*  $\chi'(G)$  is the smallest number of colors needed to color its edges so that adjacent edges (those with a vertex in common) are colored differently; for example, if  $G$  is an even cycle then  $\chi'(G) = 2$ , and if  $G$  is an odd cycle then  $\chi'(G) = 3$ . For complete graphs,  $\chi'(K_n) = n - 1$  if  $n$  is even and  $\chi'(K_n) = n$  if  $n$  is odd, and for complete bipartite graphs,  $\chi'(K_{r,s}) = \max(r, s)$ . We note that if the maximum vertex-degree in  $G$  is  $\Delta$ , then  $\chi'(G) \geq \Delta$ ; for example, if  $G$  is the Petersen graph, which is regular of degree 3, then  $\chi'(G) \geq 3$ . However, it has no 3-edge-coloring, as Julius Petersen explained when he presented it in 1898 [32]. For, suppose that there are just three colors, 1, 2, 3. Then any coloring of the outside pentagon must have the form 1, 2, 1, 2, 3 (for some permutation of the numbers, and taking account of symmetry). This forces the colors on the spokes, as shown in Figure 1. But then the two bold edges must both be colored 2, which is forbidden. So an extra color is needed, and the chromatic index is easily seen to be 4.

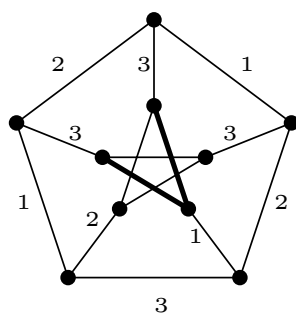


Figure 1: Edge-coloring the Petersen graph.

## 2. Early history

Like much of graph theory, edge-colorings have their origin in the four-color problem, which asks whether every map can be colored with four colors so that adjacent countries are colored differently. This problem was first posed by Francis Guthrie in 1852, and in 1879 Alfred Kempe produced a “proof” [27]. His arguments, although fallacious, contained several important ideas that would feature in the eventual proof by Kenneth Appel and Wolfgang Haken one hundred years later, and the method of Kempe chains remains a key ingredient in the theory of edge-colorings.

\*This paper is dedicated to the memory of Frank Harary.

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## 2.1 Peter Guthrie Tait

Kempe's error was not discovered until 1890, but in the meantime the mathematical physicist, P. G. Tait of Edinburgh, was convinced that Kempe's arguments could be improved. In 1880, restricting himself to cubic maps as we may (this had been shown by Arthur Cayley), he showed how a 4-coloring of the countries of a cubic map leads to a 3-coloring of the edges (a *Tait coloring*), and conversely [39]. For this, he colored the countries  $A, B, C, D$ , and then colored with color  $\alpha$  any edges between countries colored ( $A$  and  $B$ ) or ( $C$  and  $D$ ), with color  $\beta$  any edges between countries colored ( $A$  and  $C$ ) or ( $B$  and  $D$ ), and with color  $\gamma$  any edges between countries colored ( $A$  and  $D$ ) or ( $B$  and  $C$ ), giving a 3-coloring of the edges (see Figure 2). Moreover, the process is reversible – any 3-coloring of the edges leads to a 4-coloring of the countries.

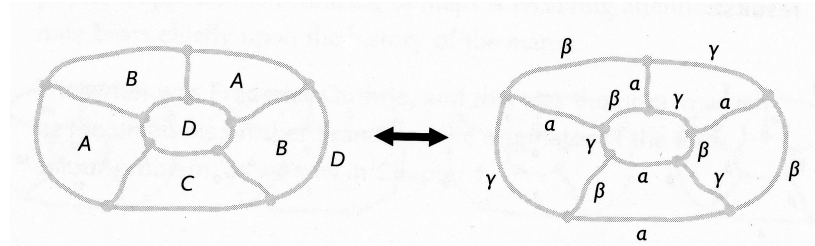


Figure 2: 4-coloring the countries of a cubic map is equivalent to 3-coloring the edges.

Tait believed that he could easily show by induction that all cubic maps can be 3-edge-colored – but he was wrong; proving this is as difficult as proving the four-color theorem. Over the years he produced further proofs of the four-color theorem, but all were incorrect.

## 2.2 Dénes König

The next edge-colorer was the Hungarian Dénes König, who wrote a lengthy paper in 1916 (in German and Hungarian) which included a section on bipartite graphs [29]. After proving that a graph is bipartite if and only if every cycle has even length, he showed that every  $k$ -regular bipartite graph splits into 1-factors – that is, into sets of disjoint edges that meet all the vertices; to do so, he used a Kempe-chain argument, looking at two-colored paths and interchanging the colors. If each 1-factor is given a different color we obtain a  $k$ -coloring of the edges of the bipartite graph, and it follows that the edges of any bipartite graph with maximum degree  $\Delta$  can be colored with  $\Delta$  colors. Figure 3 presents a 4-coloring of the edges of the complete bipartite graph  $K_{4,3}$  with maximum degree 4.

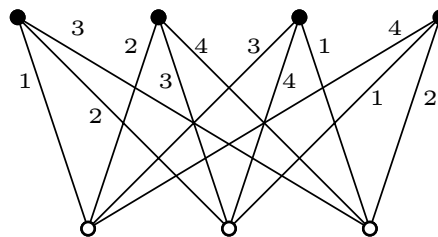
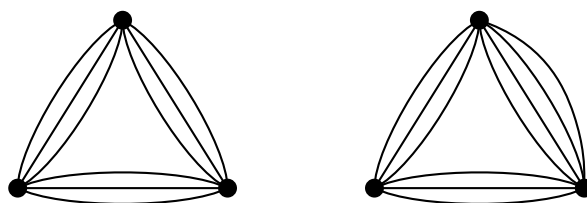


Figure 3: Edge-coloring the complete bipartite graph  $K_{4,3}$ .

Compared with publications on map coloring, which appeared with great regularity, there were initially few papers that related to coloring a graph's vertices or edges. Notable exceptions were papers on vertex-coloring by Hassler Whitney in 1932 and Leonard Brooks in 1941 [44]; the latter contained what is now known as *Brooks's theorem*, that the vertices of a connected graph with maximum degree  $\Delta$  can be properly colored with  $\Delta$  colors, except for complete graphs and odd cycles.

## 2.3 Claude Shannon

The next edge-coloring paper arose in the context of electrical networks. Claude Shannon discussed a problem on the color-coding of wires in an electrical unit, such as a relay panel, where the emerging wires at each point should be colored differently so that they can be easily distinguished. Shannon's main theorem, published in 1949 [37], was that the lines of any network can be properly colored with  $\lceil 3m/2 \rceil$  colors, where  $m$  is the largest number of wires at any point; Figure 4 shows networks where this bound is attained. Equivalently, if  $G$  is a multigraph with maximum degree  $\Delta$ , then  $\Delta \leq \chi'(G) \leq \lceil 3\Delta/2 \rceil$ ; these bounds are best possible for all values of  $\Delta$ .

Figure 4: Shannon graphs with  $\Delta = 6$  and  $\Delta = 7$ .

## 2.4 Vadim Vizing

We come now to some important articles by our next edge-colorer, Vadim Vizing. In a classic paper of 1964, written in Russian [41], Vizing obtained another upper bound for the chromatic index of a multigraph by proving that if the largest number of parallel edges in a multigraph  $G$  with maximum degree  $\Delta$  is  $\mu$ , then  $\chi'(G) \leq \Delta + \mu$  – this result is usually better than Shannon’s. In particular, if  $G$  is simple, so that there are no parallel edges, then  $\mu = 1$  and  $\chi'(G) = \Delta$  or  $\Delta + 1$ . These extremely sharp bounds led to the classification problem of deciding which simple graphs can be edge-colored with  $\Delta$  colors (such as bipartite graphs), and which ones (such as the Petersen graph) need an extra color. Vizing’s theorem for both graphs and multigraphs was independently obtained around the same time by Ram Prakash Gupta [17], whom we meet again later.

In the following year Vizing produced two further papers which, like their predecessor, completely transformed the subject [42]. Here he proved the earlier results of König and Shannon and then showed that the edges of every simple planar graph with  $\Delta \geq 8$  can be colored with  $\Delta$  colors. Some years later, this bound was reduced to 7 by three independent authors [15], and only the case of planar graphs with  $\Delta = 6$  remains unsolved.

In his papers Vizing introduced the crucial idea of a *critical graph* – one that “only just” needs the extra color, in the sense that the deletion of any edge lowers the chromatic index; such graphs have more structure than graphs in general and are therefore more susceptible to investigation. He also proved what came to be known as *Vizing’s adjacency lemma*, which asserts that, in such simple graphs, the vertices of maximum degree appear throughout the graph. In particular, if  $v_1$  and  $v_2$  are any two adjacent vertices with  $\deg(v_1) = k$ , then, if  $k < \Delta = \chi'(G) - 1$ ,  $v_2$  is adjacent to at least  $\Delta - k + 1$  vertices of degree  $\Delta$ , and if  $k = \Delta$ , then  $v_2$  is adjacent to at least two vertices of degree  $\Delta$ . It follows that every critical graph has at least three vertices of maximum degree. Vizing’s adjacency lemma has proved to be extremely useful in a number of contexts.

Around this time Vizing also extended the concept of edge-coloring to *total coloring*, where one colors both the vertices and the edges of a graph with adjacent and incident vertices and edges colored differently. This idea had independently been examined by Mehdi Behzad at Michigan State University and first appeared in written form in his Ph.D. thesis of 1965 [4]. Both Behzad and Vizing [43] proposed the *total coloring conjecture* that the vertices and edges of every simple graph  $G$  have a total coloring in at most  $\Delta + 2$  colors. This remains unproved.

### *Robin Wilson writes:*

In 1969 Frank Harary produced his classic book, *Graph Theory* [22]. Around this time I had just taken up graph theory and was teaching it at Oxford University while beginning to write my *Introduction to Graph Theory*, which was first published in 1972 [45].

While reading Harary’s book, I came across his statement of Vizing’s theorem, which was followed by the remark: “It is not known in general for which graphs  $\chi' = \Delta$ ”. This sentence changed my life! I became interested in edge-colorings, and shortly after this met Lowell Beineke with whom I wrote a joint paper on the subject [6]. Here we adopted my suggestion to saying that a graph is of class one if  $\chi' = \Delta$  and of class two if  $\chi' = \Delta + 1$  – terminology that is now standard. This paper gave constructions for graphs of class 2 and seems to have sparked an interest in the subject throughout the 1970s.

Shortly after this I relocated to the Open University, where I was to work for 37 years. Encouraged to supervise a doctoral student, I invited Stanley Fiorini, a Maltese graduate to whom I had taught graph theory in Oxford, to work on edge-colorings of graphs. He made rapid progress, developing the work of Vizing with new results on the chromatic index of planar graphs, uniquely edge-colorable graphs, the number of edges in critical graphs, and much else besides. His thesis, submitted after just two years, was highly praised by the external examiner, Richard Rado, and was awarded the Open University’s first Ph.D. degree in pure mathematics. Shortly afterwards, Stanley Fiorini and I collaborated on the book, *Edge-colourings of graphs* [14], which was based on his thesis and appeared in 1977.

Following the publication of our book, Fiorini published a comprehensive bibliographic survey of edge-coloring [13], where he pointed out that “most of the work that has been done since 1964 has been concerned with the classification



Figure 5: Frank Harary and Robin Wilson, Balatonfüred, Hungary, 1969; photograph by Bjarne Toft.

problem”. Shortly after this I achieved Erdős number 1 when I collaborated with Paul Erdős on a short note [11] that showed that almost all graphs are of class one, in the sense that if  $P(n)$  is the probability that a random graph with  $n$  vertices is of class 1, then  $P(n) \rightarrow 1$  as  $n \rightarrow \infty$ .

Much of the edge-coloring activity during the 1970s was devoted to finding a proof of the *critical graph conjecture*, that there are no simple critical graphs of class two with an even number of vertices. Versions of this conjecture appeared in my paper with Lowell Beineke, and independently in the writings of Ivan T. Jakobsen [24] who proved that there are no critical graphs of orders 4, 6, 8, and 10. Subsequently, Beineke and Fiorini [5] showed that there are no critical graphs with  $\chi' = 4$  and 12 vertices. Lars Andersen then extended this to graphs with 14 vertices and (with Fiorini) to 16 vertices [2].

Meanwhile I had also become interested in *snarks*. These are cubic graphs of class two and were so christened by Martin Gardner after Lewis Carroll’s *The Hunting of the Snark*, because they are hard to find. The smallest snark is the Petersen graph, and until 1973 only three others were known. But in the 1970s, some infinite families of snarks were obtained by Rufus Isaacs, Emanuel Grinberg, Feodor Loupekhine, and others (see [10]). Imagine my surprise, then, when in early 1979 Mark Goldberg of Novosibirsk sent me the letter reproduced in Figure 6. Here he describes how he had just constructed a new infinite family of snarks with the extra property that the removal of just two edges from each one yields a counterexample to the critical graph conjecture, which was therefore false [20].

We conclude this section with a major result. In 1966 W. T. Tutte conjectured that every snark includes the Petersen graph as a minor, and a proof of this was announced in 2001 by Neil Robertson, Daniel P. Sanders, Paul Seymour, and Robin Thomas.

### 3. The modern history

Until the 1970s simple graphs had received most attention in the literature, but multigraphs had not been entirely ignored, as we now discover.

#### 3.1 Mark Goldberg

In 1973, in Akademgorodok outside Novosibirsk, Mark Goldberg published a paper [19] in the journal where Vizing had published, where he made a conjecture on the chromatic index of a multigraph  $G = (V(G), E(G))$ . He had noticed that a lower bound for  $\chi'(G)$  is  $|E(G)|/\lceil |V(G)|/2 \rceil$ , and he deduced that  $\chi'(G) \geq \max_H \lceil |E(H)|/\lceil |V(H)|/2 \rceil \rceil = w(G)$ , where the maximum is taken over all induced submultigraphs  $H$  of  $G$  with an odd number of vertices. The parameter  $w(G)$  is called the *density* of  $G$ . Goldberg made the following conjecture:

**Goldberg’s conjecture:**  $\chi'(G) = \Delta(G)$ ,  $\Delta(G) + 1$ , or  $w(G)$ .

This means that if  $w(G) \leq \Delta(G)$ , then  $\chi'(G)$  is  $\Delta(G)$  or  $\Delta(G) + 1$ , and if  $w(G) \geq \Delta(G) + 1$ , then  $\chi'(G) = w(G)$ . It follows



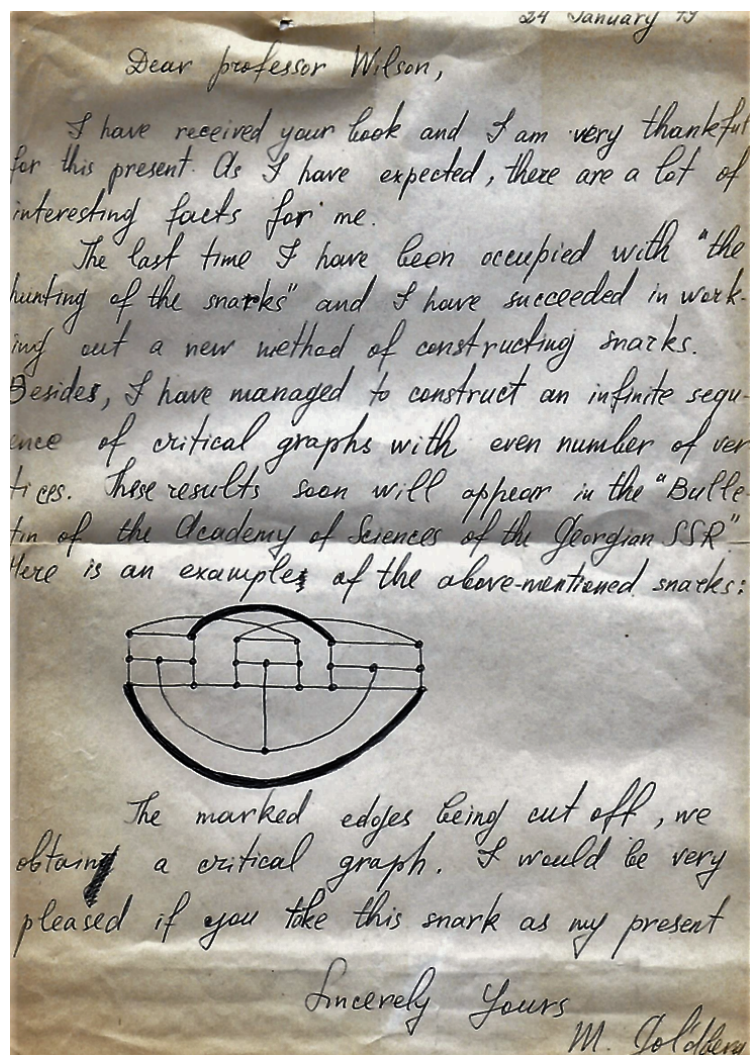


Figure 6: A letter from Mark Goldberg to Robin Wilson, January 1979.

that Goldberg's conjecture may also be expressed in the form

**Goldberg's conjecture:**  $\chi'(G) = \max\{\Delta(G), w(G)\}$  or  $\Delta(G) + 1$ .

The importance of the latter formulation is that both of the two numbers are polynomially computable. For the first number, this was shown in a groundbreaking paper by Paul Seymour [35], via the fractional chromatic index and Edmonds' matching polytope, and so the difficulty of computing  $\chi'(G)$  for a multigraph  $G$  is "only" that of distinguishing between the cases  $\Delta(G)$  and  $\Delta(G) + 1$ , and this is an NP-complete problem, even for  $\Delta(G) = 3$ , as shown by Ian Holyer [23]. It seems to be unknown whether  $w(G)$  is polynomially computable when  $w(G) \leq \Delta(G)$ . We also note that Goldberg's conjecture implies the critical graph conjecture for multigraphs with  $\chi'(G) \geq \Delta(G) + 2$ .

### 3.2 Paul Seymour

Similar ideas, and the same conjecture, were formulated by Seymour, and it is now often called the *Goldberg-Seymour conjecture*. In 1974 he submitted his above-mentioned comprehensive paper, published only in 1979, and was the first to emphasize the importance of the density as an interesting graph parameter by itself. Seymour also presented the conjecture at a conference in Waterloo in 1977 and it was published in its proceedings two years later [36]. He also formulated other conjectures, such as the following one:

**Seymour's exact conjecture:** If  $G$  is a planar multigraph, then  $\chi'(G) = \Delta(G)$  or  $w(G)$ .

### 3.3 Ram Prakash Gupta

The Goldberg conjecture was also proposed independently by R. P. Gupta. It was presented at a conference in Kalamazoo in 1976 and appeared in its proceedings in 1978 [16]. Gupta had come to the United States from India, where he had independently obtained Vizing's theorem for graphs and multigraphs, as we mentioned earlier. Gupta also gave an extended

parameterized generalization of the conjecture which turns out to be equivalent to it. Suppose that  $G$  is a multigraph with  $\chi'(G) = w(G)$ . Then  $G$  contains a submultigraph  $H$  with  $2t + 1$  vertices for which

$$\chi'(G) = \lceil |E(H)|/t \rceil \leq \Delta(G)(2t + 1)/2t + (t - 1)/t = \Delta(G) + 1 + (\Delta(G) - 2)/2t.$$

It follows from this that Goldberg’s conjecture is equivalent to the following statement:

**Gupta’s conjecture:** *If  $\chi'(G) > \Delta(G) + 1 + (\Delta(G) - 2)/2t$  for a fixed number  $t \geq 1$ , then  $\chi'(G) = w(G)$  and  $G$  contains a submultigraph  $H$  with  $2t' + 1$  vertices for which  $\chi'(G) = \lceil |E(H')|/t' \rceil$ , where  $1 \leq t' < t$ .*

For  $t = 1$  this conclusion is always false, and so the premise is also; this is Shannon’s theorem. For  $t = 2$  the statement is that if  $\chi'(G) > 5\Delta(G)/4 + 1/2$ , then  $\chi'(G) = w(G)$  and  $G$  contains three vertices joined by  $\chi'(G)$  edges; this was first proved by Goldberg [19], and independently by Andersen [1] and Gupta [16], who both also obtained the case  $t = 3$ ; the case  $t = 2$  had already appeared as an exercise in the Fiorini-Wilson book [13] with a reference to Goldberg. Goldberg [21] also settled the case  $t = 4$ , and this whole development over seventeen years ended in 1990 with the 1.1–theorem of Nishizeki and Kashiwagi [31]:

**The 1.1–theorem:** *If  $\chi'(G) > 1.1\Delta(G) + 0.8$ , then  $\chi'(G) = w(G)$ .*

Let us remark that Andersen [1] also formulated a conjecture equivalent to Goldberg’s conjecture, see [38].

### 3.4 Vladimir Aleksandrovich Tashkinov

Ten years later a breakthrough appeared, again in Akademgorodok, when V. Tashkinov [40] presented a greatly improved proof of the 1.1–theorem, based on what we now call *Tashkinov trees*. This sparked renewed interest, with proofs of the various cases of Gupta’s conjecture up to  $t = 7$  (see [38]; this last case was first obtained by Diego Scheide in his doctoral thesis, supervised by Michael Stiebitz, and was published in [34]). The culmination came in 2018 with a long and complicated paper by Chen, Jing, and Zang [8], proposing a proof of the full Goldberg-Seymour conjecture, while Scheide had earlier proved a weaker form:  $\chi'(G) \leq \max\{\Delta(G) + \sqrt{(\Delta(G) - 1)/2}, w(G)\}$ .

To understand the importance of Tashkinov trees, we return to Vizing’s proof that, for a multigraph  $G$ ,  $\chi'(G) \leq \Delta(G) + \mu(G)$ ; it was based on the use of Kempe chains to recolor the edges of multi-fans. Let  $e_1 = xy_1$  be an edge of  $G$ , and let  $\varphi$  be an edge- $k$ -coloring of  $G - e_1$ . A *fan*  $F$ , with respect to  $e_1$  and  $\varphi$ , is a sequence  $e_1, e_2, \dots, e_p$  of distinct edges, with  $e_i = xy_i$  and where, for all  $i \geq 2$ , the color  $\varphi(e_i)$  is missing at a vertex  $y_j$  with  $j < i$ . (Note that if the fan has multiple edges, then the vertex-set  $W = \{y_1, y_2, \dots, y_p\}$  does not have  $p$  elements.) If  $\chi'(G) = k + 1$ , then there is a vertex  $y_2$  that differs from  $y_1$  (so  $|W| \geq 2$ ), and the sets of missing colors at two different vertices  $y_i$  and  $y_j$  and at  $x$  and  $y_i$  are always disjoint. Moreover, if the fan cannot be extended by a further edge, then the following fan equation holds:

**The fan equation:** *If  $\chi'(G) = k + 1$ ,  $\chi'(G - xy_1) = k$ , and  $F$  is a maximal fan with respect to  $xy_1$  and a  $k$ -coloring of  $G - xy_1$ , then*

$$\sum_{z \in W} (\deg(z) + \mu^*(x, z) - k) = 2,$$

where the sum extends over the set  $W$  of all distinct vertices  $z \neq x$  in edges of  $F$ , and where  $\mu^*(x, z)$  is the number of edges in  $F$  joining  $x$  and  $z$ .

The fan equation has many direct consequences. Because there must be at least one positive term in the sum,  $x$  has a neighbor  $z$  for which

$$\chi'(G) = k + 1 \leq \deg(z) + \mu^*(x, z) \leq \Delta(G) + \mu(G),$$

which is Vizing’s theorem. Note that the two largest terms in the sum must add up to at least 2, and so  $x$  has two different neighbors  $z_1$  and  $z_2$  for which

$$2\chi'(G) = 2k + 2 \leq \deg(z_1) + \deg(z_2) + \mu^*(x, z_1) + \mu^*(x, z_2) \leq 3\Delta(G),$$

which is Shannon’s theorem. If  $G$  is simple and  $\chi'(G) = \Delta(G) + 1$ , then  $\deg(z_1) = \deg(z_2) = \Delta(G)$  – that is,  $x$  has at least two neighbors of maximum degree, which is part of Vizing’s adjacency lemma.

It is clear from this that a graph  $G$  has to fulfill rather restrictive conditions to satisfy  $\chi'(G) = \Delta(G) + \mu(G)$ . Indeed, Vizing had pointed out that there are numbers  $D$  and  $M$  for which any graph  $G$  with  $\Delta(G) = D$  and  $\mu(G) = M$  satisfies  $\chi'(G) \leq \Delta(G) + \mu(G) - 1$ . Gupta [16] had constructed graphs with  $\chi'(G) = \Delta(G) + \mu(G)$  and conjectured that such graphs  $G$  can exist only for those values of  $\Delta(G)$  and  $\mu(G)$  where his construction works. Scheide [34] proved this conjecture of Gupta under the assumption that Goldberg’s conjecture holds.

A new theory for edge-coloring was created in 1984 by Henry Kierstead [28], whose idea resembled Vizing's, but replaced fans by paths. Again,  $k$ -edge-colorings of  $G - xy$  were considered for graphs  $G$  with  $\chi'(G) = k + 1$ , with the object of finding many vertices whose missing sets of colors were all disjoint. After the results of Vizing and Kierstead it was tempting to seek a common generalization for trees, where fans and paths are two special cases. This quest is what Tashkinov carried out and his method is a powerful one, leading to many new results. His definition of a *Tashkinov tree* was analogous to that given above for Vizing fans (see [38]). Limitations were described by Asplund and McDonald [3] in 2016. Two years later Chen, Jing, and Zang [8] introduced a new type of extended Tashkinov tree for their proof of the full Goldberg-Seymour conjecture.

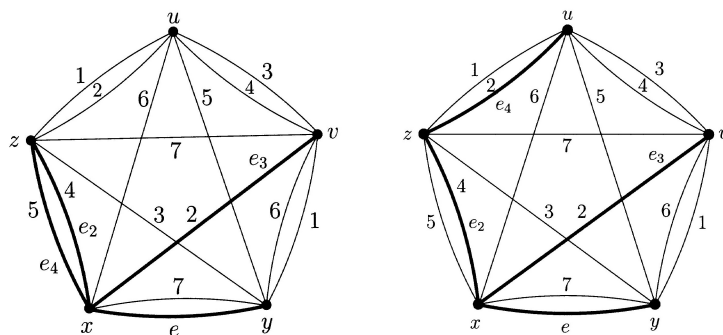


Figure 7: A Vizing-fan and a Tashkinov tree, each with four bold edges.

## 4. Some further results

Further information on edge-colorings can be found in the books by Fiorini and Wilson [14] and Stiebitz et al. [38] and in survey articles by McDonald [30] and Yan et al. [46]. Lists of unsolved problems can be found in [26, 38], and for many of these problems there is a rich literature with partial and/or related results and theories. In addition to the total coloring conjecture and other unanswered problems mentioned earlier, three intriguing questions that remain are as follows.

- The *overfull conjecture* of Anthony Hilton [9] dates from 1985 and asks whether  $\chi'(G) = w(G)$  if  $G$  is simple,  $\chi'(G) = \Delta(G) + 1$ , and  $\Delta(G) > |V(G)|/3$ .
- The *list-coloring conjecture*, posed by several authors, first appeared in print in 1985 in a paper of B. Bollobás and A. J. Harris [7]. Let  $\chi'(G) = k$ , and let each edge of  $G$  be assigned a list of  $k$  colors. Does  $G$  always have an edge-coloring in which each edge receives a color from its list? For bipartite graphs this was verified in 1995 by F. Galvin [18].
- An *interval coloring* of the edges of a graph uses colors  $1, 2, 3, \dots$  and the set of colors at each vertex are consecutive integers. A question that arises in scheduling asks for the existence of interval-colorings of bipartite bi-regular graphs, where all the vertices in each of the two partite sets have the same degree. If these degrees are  $a$  and  $b$  with  $a \leq b$ , then an interval-coloring exists when  $a = b$  (by König's theorem), and also when  $a = 2$ . The first unknown case is  $a = 3, b = 4$ , where such a coloring requires at least six colors; note that the edge-coloring in Figure 3 is not an interval coloring.

### *Bjarne Toft writes:*

First an anecdote from 1985. Frank Harary, Paul Erdős, and Jarik Nešetřil were visiting me in Odense and I took them in our small Honda Civic to see the town. All of a sudden we ran out of gas. I was perplexed – with important guests around, I did not think about such trivialities as a non-empty tank. Harary took command of the situation and within a few minutes he had stopped a taxi. We took it to the nearest gas station, bought a can and filled it, took the taxi back, and after fifteen minutes of Erdős-Nešetřil theorem-proving we were back on track. Harary also visited Denmark on later occasions, and this was as always interesting, colorful, and fun. With his enormous enthusiasm for graph theory he attracted students to the subject, while confirming the prejudices of some of my colleagues that it is not serious mathematics. On a train from Odense to Copenhagen he met a student from the Niels Bohr Institute, resulting in his giving a lecture there. When I asked him how it was, he replied: “Very Bohr-ing”.

In 1998 I supervised a Master's thesis at the University of Southern Denmark for a bright student, Lene Monrad Favrholt. Because Vadim Vizing had visited our University a couple of times in the 1990s, where Robin also met him, I naturally suggested that Lene take edge-colorings of graphs as her topic, and I proposed that we should look at Vizing's



and Kierstead’s proofs, based on different trees (fans and paths), and try to find a common generalization. This project was unsuccessful, but Lene produced the fan equation and pointed out its usefulness, implying improved proofs of known results and giving rise to some new ones. We started to work on a survey on the fan equation and its implications.



Figure 8: Vadim Vizing and Bjarne Toft at the University of Odense (now Southern Denmark), 1995.

In 1999, I visited Akademgorodok in Siberia, taking part in a fantastic meeting which Vizing (and many others) also attended. Three years later, Vizing revisited the University of Southern Denmark. With Michael Stiebitz also present, our discussions led to a proof of Hadwiger’s conjecture for line graphs of multigraphs; for simple graphs this was an easy consequence of Vizing’s theorem, but not for multigraphs. Vizing had posed the problem in 1968 (see [43]), but Bruce Reed had told me (in 2000) that he and Paul Seymour had solved it. Seymour now informed us that a paper had been written, but never submitted – but in 2004 it was published [33]. Because our proof was very attractive we hoped that it would be at least as good as the Reed-Seymour one. To our surprise, the two proofs were almost identical, including an independent statement by Reed and Seymour of the fan equation.

In 2000 Tashkinov’s method appeared, but alas in Russian. However the Technical University in Ilmenau in Thüringen, Germany, invited Tashkinov to visit there, and he explained his method in detail to Michael Stiebitz. Since the early 1990s Stiebitz had visited Odense for a month each year, and so he joined the survey-paper project, while immediately outdistancing Lene and me. In 2006 our paper was ready and appeared as a preprint [12], but the subject was far from finished. As a result, our preprint was never submitted for publication, but continued to be extended with new material. In Ilmenau Michael had a brilliant Ph.D. student, Diego Scheide, who developed his own deep insights and produced exciting new results, as we saw earlier.

In the end, in 2012, Stiebitz completed the manuscript, that had now developed into a book. We contacted Wiley, who had published an earlier book of Tommy R. Jensen and me [25] in 1995, and they were happy to take the new monograph [38]. We believe that it has made an impact, even if no review or evaluation has yet appeared. According to *Mathematical Reviews*, at least 50 papers had cited it by 2021. Many new active researchers have taken an interest in edge-colorings, and continue to develop the subject as it continues to acquire new results, consider loose ends, and pose interesting problems.

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